Solution of an initial-boundary value problem for coupled nonlinear waves

1990 J. Phys. A: Math. Gen. 23 1385

(http://iopscience.iop.org/0305-4470/23/8/013)

J. Phys. A: Math. Gen. **23** (1990) 1385-1403. Printed in the UK

Solution of an initial–boundary value problem for coupled nonlinear waves

J Leon and A Latifi

Laboratoire de Physique Mathématique†, Université des Sciences et Techniques du Languedoc, 34095 Montpellier Cedex 05, France

Received 21 October 1989

Abstract. We derive and study a hierarchy of nonlinear coupled evolution equations (among which is the coupled Korteveg-de Vries-Schrödinger equation) for which we prove that some mixed initial-boundary value problem is solvable. We give the method of solution together with the Bäcklund transformation and establish the infinite set of conserved densities. We finally discuss the applicability of such equations in plasma physics and hydrodynamics.

1. Introduction

We give the general method of solution of the system

$$\begin{aligned} &u_{xx}+\lambda^2 u = qu \qquad q=q(x,t) \qquad u=u(\lambda;x,t) \\ &q_t+6qq_x-q_{xxx} = -\frac{\partial}{\partial x}\int\int_{\mathbb{C}} \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}\; u(\lambda;x,t)u(-\lambda;x,t)\mu(\lambda,t) \end{aligned} \tag{1.1}$$

($\mu(\lambda,t)$ is an arbitrary distribution in $\lambda\in\mathbb{C}$ and $\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}=-2\mathrm{i}\mathrm{d}\lambda_{\mathrm{R}}\,\mathrm{d}\lambda_{\mathrm{I}}$ for $\lambda=\lambda_{\mathrm{R}}+\mathrm{i}\lambda_{\mathrm{I}}$) when the following initial-boundary values are prescribed

$$q(x,t)|_{t=0}=q_0(x) \qquad u(\lambda;x,t)\xrightarrow[x\to+\infty]{} d(\lambda,t)\exp[-\mathrm{i}\lambda(x-\lambda^2 t)]. \tag{1.2}$$

In the reference frame (x,t), equation (1.1) represents the interaction of a wavepacket of high-frequency waves $u(\lambda;x,t)$ with the single, low-frequency wave $q(x,t)$. This equation is of general interest in physics and we discuss in section 5 some closely related equations appearing in plasma physics (coupling of a plasma wavepacket to acoustic waves) and hydrodynamics (interaction between long and short capillary-gravity waves).

We prove here that the initial-boundary value problem (1.2) for the system (1.1) can be solved by the spectral transform theory [1] within the framework of singular dispersion relations [2-7] (for $q_0(x)$ and $d(\lambda,t)$ in some spaces of functions).

A different version of (1.1) in which one chooses for $u(\lambda,x,t)$ the set of N bounded eigenfunctions ($\lambda^2=-\lambda_n^2$) (therefore setting $\mu(\lambda)=\Sigma_1^N\,\delta(\lambda-\mathrm{i}\lambda_n)\alpha_n$, α_n some constant), has been studied by Mel'nikov [8]. There it has been shown that the initial value problem is integrable by means of the usual 'Lax pair formalism' involving a matrix spectral operator of rank $N+3$ (the initial datum $q(x,0)$ should of course belong to the set of potentials having N discrete eigenvalues).

† Unité de Recherche Associée au CNRS no URA 040768.

0305-4470/90/081385+19\$03.50

Solving the initial-boundary value problem (1.2) for the system (1.1), we will derive a set of relevant properties of the nonlinearily coupled waves.

First of all, the system may exhibit (depending on the choice of the distribution $\mu(\lambda)$) the property of transparency to the waves $u(\lambda; x, t)$ coming from $x=\infty$. This property may also occur only for particular wavenumbers λ, the system then acts as a filter. The same kind of behaviour was derived for the self-induced transparency equations [9] and are a direct consequence of the non-analyticity of the dispersion relation.

Second, for the wavenumbers for which the system is 'transparent', the field q rapidly becomes a superposition of solitons, each of which is accompanied by 'a piece' of the wave u. More precisely, as $t\to\infty$, q separates into N individual solitons and the wave u into N waves, each locked to its own soliton. Such a mutual selective trapping has been observed in a different context (Zakharov equations for a plasma) but still for a similar physical situation (coupling of HF/LF waves) in [10] where 'plasmon wavepackets (u) are shown to be nucleated in narrow density holes (q)'.

Finally, although the system has an infinite sequence of conservation laws, these do not lead in general to conserved quantities. The time asymptotics result in a transfer of the energy of the wave u to the physical system (see e.g. [9]). This phenomenon can be evaluated exactly because the time dependence of what would have been the conserved quantities can be explicitly integrated.

In section 2 we establish the general hierarchy of nonlinear evolutions containing (1.1) as a special case and for which an initial-boundary value problem analogue to (1.2) is solvable. This hierarchy also contains as a special case the caviton equation [11] (a dispersionless and partially linearised version of (1.1)) which is a model whereby solitons are used to represent the electronic density depressions (cavitons) in a plasma. The method used here is that of singular dispersion relations [2-7] applied to the Schrödinger eigenvalue problem.

In section 3 we give the general method of solution and the one- and two-soliton solutions and discuss their dyanmics for some representative choices of the distribution $\mu(\lambda)$ in (1.1).

Section 4 is devoted to deriving the infinite set of conservation laws and to the discussion of the conserved (or not) quantities.

In section 5 we consider the physical situations which lead to similar systems. We show in particular that a standard situation in plasma physics does not lead to an integrable system, but rather to what we call a parametrically nearly integrable system.

Comment. Equation (1.1) (for q and λ real) can also be thought of as a 'practical tool' for the nonlinear quantum mechanics of a free particle with wavefunction u and current density q.

2. Singular general evolution equations related to the Schrödinger spectral problem

We prove in this section that the system (1.1) is integrable by establishing the general class of nonlinear evolution equations associated with the Schrödinger spectral problem for non-analytic dispersion relations. A subclass of these nonlinear evolution equations, namely the class for which the dispersion relation vanishes at infinity, has already been obtained by Kaup [11] through standard (scattering) methods.

Because it has proved [3-19] to be very useful (at least to simplify the formalism), we proceed here through the $\bar{\partial}$ problem associated with the Schrödinger spectral problem. This approach has been adopted as a basic tool to investigate hierarchies of integrable evolutions with polynomial dispersion relations in [12].

We choose here to write the $\bar{\partial}$ problem for the scalar ϕ in the form

$$\frac{\partial}{\partial\bar{\lambda}}\phi(\lambda)=\phi(-\lambda)r(\lambda) \qquad \lambda\in\mathbb{C} \tag{2.1}$$

$$\phi(\lambda)=1+\mathrm{O}(1/\lambda) \qquad |\lambda|\to\infty \tag{2.2}$$

where $r(\lambda)$ is a given distribution in $\mathbb{C}$. Our study is restricted to the case where $\phi(\lambda)$ has only simple poles (δ functions in r) or discontinuities on lines in the λ plane.

The solution of (2.1) obeying (2.2) is given by the solution of the following integral equation:

$$\phi(\lambda)=1+\frac{1}{2\mathrm{i}\pi}\iint_{\mathbb{C}}\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-\lambda}\phi(-l)r(l) \tag{2.3}$$

which leads to the asymptotic series ($\forall n$)

$$\phi(\lambda)=\sum_{j=0}^{n-1}\lambda^{-j}\phi^{(j)}+\mathrm{O}(\lambda^{-n}) \qquad \phi^{(0)}=1. \tag{2.4}$$

A parametric (x, t) dependence for ϕ is obtained by requiring the 'simplest integrable' (x, t) dependence for $r(\lambda)$:

$$\frac{\partial}{\partial x}r(\lambda)=[\alpha(\lambda)-\alpha(-\lambda)]r(\lambda) \tag{2.5}$$

$$\frac{\partial}{\partial t}r(\lambda)=[\beta(\lambda)-\beta(-\lambda)]r(\lambda). \tag{2.6}$$

In general α can be taken also as function of x and β a function of t, but this unnecessarily complicates the formalism. We have made the choice of odd coefficients in (2.5) and (2.6) because any even (regular) part could be scaled off through a gauge transformation of ϕ (if $\phi(\lambda)$ solves (2.1) then so also does $f(\lambda^2)\phi(\lambda)$).

The choice of $\alpha(\lambda)$ fixes the principal spectral problem and for $\alpha=\mathrm{i}\lambda$, the function

$$\psi(\lambda, x, t)=\phi(\lambda, x, t)\,\mathrm{e}^{-\mathrm{i}\lambda x} \tag{2.7}$$

solves the Schrödinger spectral problem

$$\psi_{xx}+(\lambda^2-q(x,t))\psi=0 \qquad \alpha(\lambda)=\mathrm{i}\lambda \tag{2.8}$$

with

$$q=-2\mathrm{i}\frac{\partial}{\partial x}\phi^{(1)}(x,t). \tag{2.9}$$

The proof of the above statement is given in [12] and is based on a comparison of the asymptotic expansion of $\psi_{xx}+\lambda^2\psi$ with that of ψ. In the present treatment it is essential to assume that $r(\lambda)$ is such that the integral equation (2.3) has a unique solution.

From (2.8) and (1.1), ψ and u solve the same scalar second-order differential equation. The function u is defined by its asymptotic behaviour (1.2) as $x \to \infty$, while ψ is determined by its behaviour in the complex λ plane. Therefore, to relate ψ and u, one should find the behaviour of ψ as $x \to \infty$, which is not possible in general (i.e. when the support of the distribution $r(\lambda)$ is not specified). However, this is possible at least when $q(x, t)$ is piecewise continuous, bounded and vanishing 'fast enough' as $|x| \to \infty$, $\forall t$. In that case we recall in the appendix that the support of $r(\lambda)$ is given by ($\lambda = \lambda_R + i\lambda_I$ and $\alpha(\lambda) = i\lambda$):

$$\mathrm{e}^{-2i\lambda x} r(\lambda) = \rho(\lambda_R)\delta^-(\lambda_I) + 2\pi \sum_{n=1}^{N} C_n \delta(\lambda - \lambda_n) \qquad \operatorname{Im} \lambda_n > 0 \tag{2.10}$$

where the distribution δ^- is defined by

$$\iint_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}\, f(\lambda)\delta^-(\lambda_I) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda_R\, f(\lambda_R - i0)$$

or

$$f(\lambda)\delta^-(\lambda_I) = \tfrac{1}{2} i f(\lambda_R - i0)\delta(\lambda_I). \tag{2.11}$$

By analogy with the language of the scattering theory for the Schrödinger operator, $\rho(\lambda_R)$ will be called the reflection coefficient, C_n the bound state coefficients and the set $\{\lambda_n;\ n = 1 \ldots N;\ \operatorname{Im} \lambda_n > 0\}$ the position of the bound states. Note that, due to (2.6), these quantities depend on t. We also have

$$q(x, t) \in \mathbb{R} \Rightarrow \operatorname{Re} \lambda_n = 0 \qquad n = 1 \ldots N \tag{2.12}$$

and λ_n^2 is a real negative eigenvalue of (2.8).

In the class of $\bar{\partial}$ factor r given by (2.10), we show in the appendix that

$$u(\lambda, x, t) = \psi(\lambda, x, t) d(\lambda, t)\, \mathrm{e}^{i\lambda^3 t}. \tag{2.13}$$

Therefore, in this context, the integral equation (2.3) furnishes a solution (u, q) of (1.1) through (2.9) and (2.13) if $\beta(\lambda)$ is chosen in such a way that the time evolution equation (2.6) implies the evolution equation (1.1).

We now relate the dispersion relation $\beta(\lambda)$ to integrable evolution equations in (x, t) space. The problem is to construct the so-called 'auxiliary spectral problem' or, more precisely, to evaluate the time dependence of ϕ induced by (2.6). This is done here by examining the analytical properties of the spectral Wronskian of Jaulent and Manna [13]:

$$\hat{W}[\psi(\lambda), F(\lambda)] = \frac{1}{2i\lambda}[\psi(\lambda)F(-\lambda) - \psi(-\lambda)f(\lambda)] \doteq b(\lambda). \tag{2.14}$$

Here we choose for $F(\lambda)$

$$F(\lambda) = \frac{\partial}{\partial t}\psi(\lambda) - \tfrac{1}{2}[\beta(\lambda) - \beta(-\lambda)]\psi(\lambda) \tag{2.15}$$

where ψ is given by the solution of (2.1) and (2.2) through (2.7) and $\beta(\lambda)$ is the dispersion relation entering in (2.6). An elementary calculation gives

$$2i\lambda \frac{\partial}{\partial \bar{\lambda}} b(\lambda) = \phi(\lambda)\phi(-\lambda) \frac{\partial}{\partial \bar{\lambda}}[\beta(\lambda) - \beta(-\lambda)] \tag{2.16}$$

(in the course of this calculation one realises that the choice of the odd coefficient $\beta(\lambda)-\beta(-\lambda)$ in (2.6) is essential).

It is possible to solve (2.16) if the behaviour of $\beta(\lambda)$ on the boundary (i.e. $|\lambda|\to\infty$) is known. We choose

$$\begin{aligned}\beta(\lambda)&=\mathrm{i}\lambda\sum_{j=0}^{n}\beta_{2j}\lambda^{2j}+\beta_{\mathrm{s}}(\lambda)\\ \beta_{\mathrm{s}}(\lambda)&=\mathrm{O}(1/\lambda)\qquad |\lambda|\to\infty.\end{aligned}\tag{2.17}$$

The polynomial part of β is taken to be odd because any even part vanishes either in (2.6) or (2.16). When the singular part $\beta_{\mathrm{s}}(\lambda)$ is absent, we recover the well known Korteveg-de Vries hierarchy of nonlinear evolution equations [1].

We insert the expansion (2.4) and (2.17) in the expression (2.14) of $b(\lambda)$ to get the following behaviour:

$$b(\lambda)=\sum_{p=0}^{n}\lambda^{2p}b_{2p}(x,t)+\mathrm{O}(1/\lambda)\qquad |\lambda|\to\infty.\tag{2.18}$$

Although they are not necessary for our task, we write down the expressions for the b_i in terms of β and ϕ:

$$b_{2p}(x,t)=\sum_{j=0}^{n-p}\beta_{2(p+j)}\sum_{l=0}^{2j}(-1)^l\phi_l(x,t)\phi_{2j-l}(x,t).\tag{2.19}$$

Now equation (2.16) can be solved, and gives

$$b(\lambda)=\sum_{p=0}^{n}\lambda^{2p}b_{2p}+\frac{1}{2\mathrm{i}\pi}\iint_{C}\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{2\mathrm{i}l}\frac{\partial}{\partial\bar{l}}[\beta_{\mathrm{s}}(l)-\beta_{\mathrm{s}}(-l)]\frac{\phi(l)\phi(-l)}{l-\lambda}\tag{2.20}$$

(note that $b(\lambda)=b(-\lambda)$ through the involution $l\to-l$ in the integral).

Alternatively, one can show that F defined in (2.15) can be expressed as

$$F(\lambda)=b(\lambda)\psi_x(\lambda)-\tfrac{1}{2}b_x(\lambda)\psi(\lambda)\tag{2.21}$$

just by replacing F and b by their definitions in terms of ψ and by using the two following important relations:

$$\psi(\lambda)\psi_x(-\lambda)-\psi_x(\lambda)\psi(-\lambda)=2\mathrm{i}\lambda\tag{2.22}$$

$$\hat{W}(\psi,F_x)=\hat{W}(\psi_x,F).\tag{2.23}$$

Finally, the nonlinear evolution equations are obtained, as usual, as the compatibility condition between the 'principal spectral problem' (2.8) and the 'auxiliary spectral problem' (2.15), (2.21) with $b(\lambda,x,t)$ given by (2.20). Identifying the coefficients of the different powers of λ^2 in the following expression of the compatability condition:

$$\left[\frac{\partial^2}{\partial x^2}+\lambda^2-q,\ \frac{\partial}{\partial t}-b\frac{\partial}{\partial x}+\frac{b_x}{2}\right]\psi=0\tag{2.24}$$

we obtain the following recursive determination of the b_j in (2.20):

$$\begin{aligned}&\frac{\partial}{\partial x}b_{2(j-1)}=\frac{1}{2}\left(q_x+2q\frac{\partial}{\partial x}-\frac{1}{2}\frac{\partial^3}{\partial x^3}\right)b_{2j}\qquad j=1\ldots n\\ &b_{2n}=\gamma_0\quad\text{(a constant)}\end{aligned}\tag{2.25}$$

and the evolution equation

$$q_t=\left(-\frac{1}{2}\frac{\partial^3}{\partial x^3}+2q\frac{\partial}{\partial x}+q_x\right)b_0-\frac{1}{\pi}\frac{\partial}{\partial x}\iint_C \mathrm{d}l\wedge\mathrm{d}\bar{l}\,\psi(l)\psi(-l)\frac{\partial}{\partial \bar{l}}\beta_s(l). \tag{2.26}$$

Finally the above evolution equation can be written in terms of the recursion operator

$$L=-\frac{1}{4}\frac{\partial^2}{\partial x^2}+q-\frac{q_x}{2}\int_x^{\infty}\mathrm{d}x' \tag{2.27}$$

as the following hierarchy ($n=0,1,\ldots$):

$$\begin{aligned}&q_t=\gamma_0 L^n q_x-\frac{\partial}{\partial x}\iint_C \mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}\;\psi(\lambda)\psi(-\lambda)\sigma(\lambda)\\&-\psi_{xx}+q\psi=\lambda^2\psi\end{aligned} \tag{2.28}$$

where the distribution $\sigma(\lambda)$ has been defined as

$$\sigma(\lambda)=\frac{1}{\pi}\frac{\partial}{\partial\bar{\lambda}}\beta(\lambda). \tag{2.29}$$

The system (2.28) has to be completed with the following boundary conditions (see the appendix):

$$q(x,t)\to 0 \qquad x\to\pm\infty \qquad \forall t\in\mathbb{R} \tag{2.30}$$

$$\psi(\lambda;x,t)\to \mathrm{e}^{-\mathrm{i}\lambda x} \qquad x\to+\infty \qquad \forall t\in\mathbb{R} \quad \lambda\in\mathbb{R}. \tag{2.31}$$

The above system (2.28) constitutes the general evolution equation integrable by means of the spectral transform when the initial datum $q(x,0)$ is given together with the boundary (2.31). If the dispersion relation β is analytic, then $\sigma\equiv 0$ from (2.29) and the hierarchy reduces to the usual Korteveg–de Vries hierarchy [1].

The system (1.1) is obtained for the following choices:

$$n=1 \qquad \gamma_0=-4 \tag{2.32}$$

$$\sigma(\lambda)=\mu(\lambda,t)d(\lambda,t)d(-\lambda,t) \tag{2.33}$$

which fixes the distribution $\sigma(\lambda)$ from the data of the boundary (1.2) and of the distribution $\mu(\lambda)$ (remember that u is related to ψ through (2.13)). Consequently the dispersion relation $\beta(\lambda)$ is obtained from (2.17) and (2.29) and has the form (note that (2.19) implies that $\beta_{2n}=b_{2n}\equiv\gamma_0$)

$$\beta(\lambda,t)=-4\mathrm{i}\lambda^3-\frac{\mathrm{i}}{2}\iint_C\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-\lambda}\mu(\lambda,t)d(\lambda,t)d(-\lambda,t). \tag{2.34}$$

If we choose $n=0$ instead (no polynomial part in $\beta(\lambda)$) and

$$\mu(\lambda,t)=\frac{\mathrm{i}}{2}\delta(\lambda_1) \tag{2.35}$$

we get the caviton equation [11], i.e.

$$q_t=-\frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\mathrm{d}\lambda|u(\lambda;x,t)|^2 \qquad u_{xx}+\lambda^2u=qu. \tag{2.36}$$

3. General method of solution, solitons

We now describe the method for solving the nonlinear coupled system (1.1) with the data

$$q_0(x) \qquad d(\lambda, t) \qquad \mu(\lambda, t) \tag{3.1}$$

and the asymptotic behaviours

$$\begin{aligned} &q_0(x) \to 0 \qquad x \to \pm\infty \\ &u(\lambda; x, t) \to d(\lambda, t) \exp[-\mathrm{i}\lambda(x - \lambda^2 t)] \qquad x \to \infty. \end{aligned} \tag{3.2}$$

While we are particularly interested in the system (1.1), it is, however, worth remarking that the following method of solution holds for *any equation in the hierarchy* (2.28) just by suitably modifying the expression (2.34) of the dispersion relation.

The method of solution proceeds along the following lines:

$$\left.\begin{aligned} &q_0(x) \xrightarrow{1} \{\psi(\lambda, x, 0), r(\lambda, x, 0)\} \\ &\{d(\lambda, t), \mu(\lambda, t)\} \xrightarrow{2} \beta(\lambda, t) \end{aligned}\right\} \xrightarrow{3} r(\lambda, x, t) \xrightarrow{4} \psi(\lambda, x, t) \begin{matrix} \overset{5}{\nearrow} q(x, t) \\ \underset{6}{\searrow} u(\lambda, x, t). \end{matrix} \tag{3.3}$$

We now describe each of these steps.

Step 1. This consists of solving, for ψ, the Schrödinger direct spectral problem (2.8) and writing it as a $\bar\partial$ problem to obtain $r(\lambda, x, 0)$ from the solution $\psi(\lambda, x, 0)$. We recall for completeness the method in the appendix and simply note that we are then able to prove the relation (2.13), and also the structure (2.10) of $r(\lambda, x, 0)$.

Step 2. This reduces to computing the integral in (2.34) with given distribution μ and function d.

Step 3. The solution results readily from (2.6) once $\beta(\lambda, t)$ is known, namely

$$r(\lambda, x, t) = r(\lambda, x, 0) \exp\left(\int_0^t \mathrm{d}t'(\beta(\lambda, t') - \beta(-\lambda, t'))\right). \tag{3.4}$$

Step 4. Here we have the solution of the inverse Schrödinger spectral problem given by the Cauchy-Green integral equation (2.3), which, for ψ, is

$$\psi(\lambda, x, t) = \exp(-\mathrm{i}\lambda x) + \frac{1}{2\mathrm{i}\pi} \iint_C \frac{\mathrm{d}l \wedge \mathrm{d}\bar l}{l - \lambda} \psi(-l, x, t) r(l, x, t) \exp[-\mathrm{i}(l + \lambda)x]. \tag{3.5}$$

We remark that from (2.5) with the choice $\alpha = \mathrm{i}\lambda$, the x dependence of r is simply

$$r(\lambda, x, t) = \tilde r(\lambda, t) \exp(2\mathrm{i}\lambda x). \tag{3.6}$$

Step 5. The solution $q(x, t)$ is obtained directly from (2.9), where we remember that $\phi^{(1)}(x, t)$ is the coefficient of $1/\lambda$ in the Laurent series for $\phi = \psi\, \mathrm{e}^{\mathrm{i}\lambda x}$.

Step 6. This is achieved by recalling (2.13):

$$u(\lambda, x, t) = \psi(\lambda, x, t) d(\lambda, t) \exp(\mathrm{i}\lambda^3 t). \tag{3.7}$$

In summary, the nonlinear initial-boundary value problem (1.1) and (1.2) has been reduced to a series of linear steps whose essential difficulties are the solution of two Volterra integral equations (equations (A3) and (A4) for step 1) and of a Cauchy-Green integral equation (equation (3.3) for step 4).

As usual in the spectral transform theory, the determination of $r(\lambda, x, 0)$ from $q(x, 0)$ is not in general explicitly possible. A weaker statement of integrability consists of saying that the integral equation (2.3) furnishes a solution of our system for any given 'good' distribution $r(\lambda, x, 0)$ and evolution equation (2.6). 'Good' means here 'such that the integral equation has a unique solution'.

A second important remark is that in the evolution equation (2.6), $r(\lambda)$ is a distribution and $\beta(\lambda)$ a non-analytic function; therefore some compatibility between the supports of r and β are required. In other words, not any distribution $\mu(\lambda, t)$ (see (2.34)) is compatible through the evolution equation (2.6) with the structure (2.10) of the distribution $r(\lambda, x, t)$. From now on we shall choose ($\lambda_{\mathrm{I}} = \mathrm{Im}(\lambda)$)

$$\mu(\lambda, t) = \mathrm{i}\nu(\lambda, t)\delta(\lambda_{\mathrm{I}}) \tag{3.8}$$

where $\nu(\lambda, t)$ is a real-valued function. We shall see in section 5 that this very choice actually corresponds to the usual physical situations. To stay closer to physical problems and simplify the formalism a bit, we also choose

$$d(\lambda, t) = d^*(-\lambda, t) \qquad \lambda \in \mathbb{R} \tag{3.9}$$

so that we obtain from (2.8) and (2.13)

$$u(-\lambda, x, t) = u^*(\lambda, x, t) \qquad \lambda \in \mathbb{R}. \tag{3.10}$$

The integrable system (1.1) then becomes

$$q_t + 6qq_x - q_{xxx} = -2\frac{\partial}{\partial x}\int_{-\infty}^{+\infty} \mathrm{d}\lambda\ \nu(\lambda, t)|u(\lambda, x, t)|^2 \qquad u_{xx} + \lambda^2 u = qu \tag{3.11}$$

for which the related dispersion relation is

$$\beta(\lambda, t) = -4\mathrm{i}\lambda^3 - \mathrm{i}\int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-\lambda}\,\nu(l, t)|d(l, t)|^2 \qquad \lambda \in \mathbb{C}. \tag{3.12}$$

Moreover, we remark that, due to (3.10), the right-hand side of (3.11) would vanish if $\nu(\lambda, t)$ was an odd function of λ. The odd part of ν being meaningless, we choose

$$\nu(\lambda, t) = \nu(-\lambda, t). \tag{3.13}$$

Now inserting the expression for β in the evolution equation (2.6), where we use the form (2.10) of $r(\lambda, x, t)$, we identify the coefficients of $\delta^-(\lambda_{\mathrm{I}})$, $\delta(\lambda - \lambda_n)$ and $\delta'(\lambda - \lambda_n)$ to obtain

$$\frac{\partial}{\partial t}\lambda_n = 0 \tag{3.14}$$

$$\frac{\partial}{\partial t}C_n(t) = C_n(t)\left(-8\mathrm{i}\lambda_n^3 - 2\mathrm{i}\int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{l-\lambda_n}\,\nu(l, t)|d(l, t)|^2\right) \tag{3.15}$$

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\lambda, t) = \rho(\lambda, t)&\left(-8\mathrm{i}\lambda^3 - 2\mathrm{i}\,⨍_{-\infty}^{+\infty}\frac{\mathrm{d}l}{l-\lambda}\,\nu(l, t)|d(\lambda, t)|^2\right)\\ &-2\pi\rho(\lambda, t)\nu(\lambda, t)|d(\lambda, t)|^2 \qquad \lambda \in \mathbb{R}.\end{aligned} \tag{3.16}$$

To obtain the important formula (3.16), we use the Sokhotski theorem

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-\lambda} f(l)|_{\lambda=\lambda_R \pm i0} = \fint_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-\lambda_R} f(l) \pm i\pi f(\lambda_R) \tag{3.17}$$

where the slashed integral stands for the Cauchy principal value integral. It is worth remarking that, while the coefficient $\beta(\lambda)-\beta(-\lambda)$ in (2.6) is obviously an odd function of λ, this is not the case for the quantity $(\beta(\lambda)-\beta(-\lambda))\delta^-(\lambda_I)$ appearing in the evolution of $\rho(\lambda, t)$, $\lambda \in \mathbb{R}$.

From the quation (3.16) we obtain that the reflection coefficient $\rho(\lambda, t)$ ($\lambda \in \mathbb{R}$) will experience an exponential growth or damping, depending on the sign of $\int^t \nu(\lambda, t)|d(\lambda, t)|^2 \,\mathrm{d}t$. In the case when it is positive any initial condition $q_0(x)$ will rapidly evolve into a pure N-soliton solution; this phenomenon is referred to as self-induced transparency. If $\int_0^t \nu|d|^2 \,\mathrm{d}t$ is negative then ρ grows in time; in this case the reconstructed potential no longer belongs to the class of potentials for which the spectral transform is well defined [1]. This requires a completely different study, such as that of Manakov [14] for the Zakharov-Shabat spectral problem. We will not consider this problem here.

We can now construct some explicit solutions of the system (3.11) by choosing the distribution $r(\lambda, x, t)$ such as to make the integral equation (2.3) explicitly solvable. This is the case, for instance, when only the discrete spectrum is present, i.e. when

$$\rho(\lambda, t) = 0 \tag{3.18}$$

and we obtain the N-soliton solution. For $N=1$ and $\lambda_1 = ip$, we get the solution

$$q(x, t) = \frac{-2p^2}{\cosh^2[p(x-\xi(t))]} \tag{3.19}$$

$$u(\lambda, x, t) = d(\lambda, t)\exp(-i\lambda x + i\lambda^3 t)\left(1 + \frac{ip}{\lambda - ip}\frac{\exp[-p(x-\xi(t))]}{\cosh[p(x-\xi(t))]}\right) \tag{3.20}$$

$$\xi(t) = \frac{1}{2p}\ln\frac{C_1}{2p} = \xi(0) - 4p^2 t - \alpha(t) \tag{3.21}$$

$$\alpha(t) = \frac{i}{p}\int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{l-ip}\,\nu(l, t')|d(l, t')|^2. \tag{3.22}$$

The only difference from the usual Kortveg-de Vries soliton comes from the term $\alpha(t)$ in the position $\xi(t)$ of the soliton (note that to ensure the regularity of q, α has to be real). This means that the result of the coupling between q and u is to modify the *soliton dynamics* (when no radiation is present). In particular the soliton can be driven by varying in time the amplitude $d(\lambda, t)$ of the applied field $\mu(\lambda, x, t)$.

Following [1], the two-soliton solution can be written ($\lambda_j = ip_j$, $j=1,2$, $p_2 > p_1$):

$$q(x, t) = -\frac{\partial}{\partial x}\frac{A_1 + A_2 + A_1 A_2 (p_1+p_2)^{-1}}{1 - \frac{1}{4}A_1 A_2 (p_1+p_2)^{-2}} \tag{3.23}$$

$$u(\lambda, x, t) = -\frac{i}{\lambda - ip_2}\left(u_1(\lambda, x, t)\frac{\partial}{\partial x}\ln F - \frac{\partial}{\partial x}u_1(\lambda, x, t)\right) \tag{3.24}$$

with the following definitions:

$$A_j = \frac{-4p_j}{1+\exp[2p_j(x-\xi_j)]} \qquad j = 1, 2 \tag{3.25}$$

$$F=\exp[p_2(x-\xi_2)]\left(1+\frac{p_1}{p_2-p_1}\frac{\exp[-p_1(x-\xi_1)]}{\cosh[p_1(x-\xi_1)]}\right)$$
$$+\exp[-p_2(x-\xi_2)]\left(1-\frac{p_1}{p_2+p_1}\frac{\exp[-p_1(x-\xi_1)]}{\cosh[p_1(x-\xi_1)]}\right). \qquad (3.26)$$

In the above formulae, ξ_j $(j=1,2)$ is given by (3.21) and (3.22) where p is to be replaced with p_j, so as for $u_j(\lambda, x, t)$ given by (3.20).

It is well known that asymptotically in time $q(x, t)$ separates into two single solitons with parameters p_1 and p_2. This is also the case for the wave $u(\lambda, x, t)$. Indeed we easily obtain from (3.24) that, for instance,

$$u(\lambda, x, t)\to u_2(\lambda, x, t) \quad \text{for } \xi_1\to-\infty \text{ with } x-\xi_2 \text{ fixed.} \qquad (3.27)$$

The other limits are obtained in the same way with the additional aid of the permutability theorem which can be stated here by saying that (3.24) is invariant under the exchange $1\leftrightarrow 2$.

This asymptotic property of mutual selective trapping holds for an N-soliton solution: each soliton component of $q(x, t)$ eventually travels locked to its own eigenmode. It is a general property of integrable systems of nonlinear coupled waves; it also appears in non-integrable systems like the Zakharov equation for plasmas where the 'nucleation of the plasma wave' was proved on the basis of numerical simulations [11].

4. Conservation laws

It is well known that an integrable evolution equation has an infinite sequence of conservation laws when the dispersion relation is polynomial in the spectral parameter λ [1]. Moreoever, when the field vanishes asymptotically the conservation laws give rise to infinitely many conserved quantities.

The situation is different when the dispersion relation is not analytic everywhere, and we will prove below that one can still write an infinite sequence of conservation laws related to (3.11) (or (1.1)). However, these do not in general lead to conserved quantities. This situation is similar to that in [9].

We shall not give many details and refer to [1] for the explicit computation. The method consists of performing an asymptotic expansion of the equation of conservation

$$y_t=\eta_x \qquad (4.1)$$

in powers of the small parameter

$$\varepsilon=1/2p \qquad (4.2)$$

where p is the parameter of the Bäcklund transformation relating two solutions (q, u) and (q', u') of (3.11) or (1.1):

$$q=-w_x \qquad q'=-w'_x \qquad w'_x+w_x=-\tfrac{1}{2}(w'-w)(4p+w'-w) \qquad (4.3)$$

$$u'(\lambda)=\frac{-\mathrm{i}}{\lambda-\mathrm{i}p}[u(\lambda)v-u_x(\lambda)] \qquad v=\frac{\partial}{\partial x}\ln[2pu(\mathrm{i}p)+cu(-\mathrm{i}p)] \qquad (4.4)$$

where c is a real constant.

In (4.1) we set

$$y=p(w'-w) \qquad (4.5)$$

and compute η. To this end it is necessary to prove the following essential property of the 'squared eigenfunction' $u(\lambda)u(-\lambda)$ for (1.1) (or $|u(\lambda)|^2$ for (3.11)) when u and u' are related by (4.4):

$$u'(\lambda)u'(-\lambda)-u(\lambda)u(-\lambda)=\frac{1}{2}\frac{\partial}{\partial x}\frac{1}{\lambda^2+p^2}\left(\frac{\partial}{\partial x}-2v\right)u(\lambda)u(-\lambda). \tag{4.6}$$

The above equation is proved by using (4.4) and the equation for u (and u') in (1.1).

Then we may prove that (4.1) with (4.5) holds for the following choice of η:

$$\eta=y_{xx}-3y^2-2\varepsilon^2y^3+\frac{1}{4\varepsilon}(\partial_x-2v)\iint_{\mathbb{C}}\frac{\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}}{\lambda^2+p^2}u(\lambda)u(-\lambda)\mu(\lambda,t) \tag{4.7}$$

when (u, q) and (u', q') are the solutions of (3.11) related by the Bäcklund transformation.

We shall use the above equation with the expansions

$$y=\sum_{m=0}^{\infty}y^{(m)}\varepsilon^m \qquad \eta=\sum_{m=0}^{\infty}\eta^{(m)}\varepsilon^m \tag{4.8}$$

where the $y^{(m)}$ are obtained from (4.3) and (4.5)

$$y^{(0)}=q \qquad y^{(1)}=-q_x \qquad y^{(m+1)}=-y_x^{(m)}-\sum_{j=0}^{m-1}y^{(j)}y^{(m-j-1)}. \tag{4.9}$$

To compute the $\eta^{(m)}$ it is necessary to know the expansion of v and of the integral on the right-hand side of (4.7). By definition, v satisfies

$$v_x+v^2=-w_x+\frac{1}{4\varepsilon^2} \tag{4.10}$$

and therefore

$$\begin{aligned}&v=\frac{1}{2\varepsilon}\sum v^{(m)}\varepsilon^m \qquad v^{(0)}=1 \qquad v^{(1)}=0\\&v^{(2)}=-2w_x \qquad v^{(m)}=-\frac{1}{2}\sum_{j=1}^{m-1}v^{(j)}v^{(m-j)}-v_x^{(m-1)}.\end{aligned} \tag{4.11}$$

We also have

$$\begin{aligned}&\frac{1}{4\varepsilon}\iint_{\mathbb{C}}\frac{\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}}{\lambda^2+p^2}u(\lambda)u(-\lambda)\mu(\lambda,t)=\sum_{m=0}^{\infty}\varepsilon^mN^{(m)}\\&N^{(0)}=0 \qquad N^{(1)}=\iint_{\mathbb{C}}\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}\,u(\lambda)u(-\lambda)\mu(\lambda,t)\\&N^{(2j)}=0 \qquad N^{(2j-1)}=\iint_{\mathbb{C}}\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}(-4\lambda^2)^ju(\lambda)u(-\lambda)\mu(\lambda,t).\end{aligned} \tag{4.12}$$

Equation (4.7) now furnishes the explicit values of $\eta^{(m)}$ and the evolution equation (1.1) has the following set of conservation laws:

$$\frac{\partial}{\partial t}y^{(m)}=\frac{\partial}{\partial x}\eta^{(m)} \qquad m=0,1,\ldots \tag{4.13}$$

where the $y^{(m)}$ are given in (4.9) and where

$$\eta^{(0)} = q_{xx} - 3q^2 + \iint_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}\; u(\lambda)u(-\lambda)\mu(\lambda, t) \qquad \eta^{(1)} = -\frac{\partial}{\partial x}\eta^{(0)}$$

$$\eta^{(m)} = y^{(m)}_{xx} - 3\sum_{j=0}^{m} y^{(j)}y^{(m-j)} - 2\sum_{k=0}^{m-2}\sum_{j=0}^{m-k-2} y^{(k)}y^{(j)}y^{(m-k-j-2)} + \frac{\partial}{\partial x}N^{(m)} - \sum_{j=0}^{m} v^{(j)}N^{(m+1-j)}. \tag{4.14}$$

The first conservation law ($m = 0$ in (4.13)) is simply the evolution equation (1.1).

Usually one simply integrates (4.13) on $x \in \mathbb{R}$ to get, if q vanishes as $x \to \pm\infty$, an infinite sequence of conserved quantities. This is not the case here because $u(\lambda, x, t)$ which appears in the $\eta^{(m)}$ does not vanish as $x \to \pm\infty$. Actually from (2.13) and (2.10) for $\lambda \in \mathbb{R}$, we have

$$u(\lambda, x, t) \to \begin{cases} d(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda x} & x \to +\infty \\ d(\lambda) \left(\dfrac{1}{T(-\lambda)}\mathrm{e}^{-\mathrm{i}\lambda x} - \dfrac{\rho(\lambda)}{T(\lambda)}\mathrm{e}^{\mathrm{i}\lambda x}\right) & x \to -\infty \end{cases} \tag{4.15}$$

where $T(\lambda)$ is the so-called transmission coefficient which can be explicitly constructed from the data of $\{\rho(\lambda), C_n, \lambda_n, n = 1 \ldots N\}$ [1], with help of the unitarity relation

$$T(\lambda)T(-\lambda) + \rho(\lambda)\rho(-\lambda) = 1. \tag{4.16}$$

We may now integrate (4.13) when q vanishes as $x \to \pm\infty$ (note that all $v^{(j)}$ vanish with q except $v^{(0)} = 1$):

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} y^{(2l)}(x, t)\,\mathrm{d}x = -N^{(2l+1)}\Big|_{-\infty}^{+\infty} \tag{4.17}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} y^{(2l+1)}(x, t)\,\mathrm{d}x = 0. \tag{4.18}$$

It can be shown that $y_t^{(2l+1)}$ can be expressed in terms of y_{tx}^{2j}, $j \leqslant m$, and therefore (4.18) does not represent a constant of the motion. The right-hand side of (4.17) can be evaluated in terms of $\rho(\lambda)$ by using $u(\lambda)u(-\lambda) = d(\lambda)d(-\lambda)\psi(\lambda)\psi(-\lambda)$, $\lambda \in \mathbb{R}$:

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} \mathrm{d}x\, y^{(2l)}(x, t) = \iint_{\mathbb{C}} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}\; d(\lambda)d(-\lambda)(-4\lambda^2)^l \frac{\rho(\lambda)\rho(-\lambda)}{1 - \rho(\lambda)\rho(-\lambda)}\mu(\lambda, t) \tag{4.19}$$

and it is clear that if the solution q is a pure N-soliton (i.e $\rho \equiv 0$) then the integrals of the densities $y^{(2l)}$ are effectively constants of the motion.

In physical situations, radiation is always present and the right-hand side of (4.19) does not vanish.

5. Related physical models

We wish to spend some time here deriving the basic equations in the physical situation of the coupling of plasma waves to acoustic waves. Doing this we have two aims: first we show that the equation governing this phenomenon is actually non-integrable and that this is due to the nonlinear coupling; secondly we explicitly obtain the integral term in the right-hand side of (1.1) (see (5.1*b*) below) by considering a plasma wavepacket instead of a single monochromatic wave.

Evolution problems resulting from the coupling of high-frequency waves (plasma waves) to low-frequency waves (electron-acoustic waves) in a plasma have been widely studied in recent years [15]. On a physical level the main consequence of this coupling is to make possible an energy transfer from the long-wavelength domain to the dissipative short-wavelength range [16]. This transfer is accompanied by depressions in the electronic density (cavitons) whose dynamics in one dimension are usually described in the context of soliton theory [11, 17, 18].

Many studies proceed by considering HF and LF waves separately. On the one hand the long-wavelength stationary limit for solitary Langmuir waves appears to be described by the nonlinear Schrödinger equation [16]. On the other hand, the description of the propagation of ion-acoustic waves of small amplitude in a cold-ion plasma is described by another integrable model, the Korteveg-de Vries equation [19].

Although these models account for purely nonlinear dynamical effects (solitons), they do not describe the coupling of HF waves to LF waves.

A first step in finding an integrable model describing the nonlinear coupling has been achieved by Karpman [11], who obtained an equation later generalised by Kaup [20]. This model is dispersionless and though it accounts well for the creation of cavitons through nonlinear energy transfer, it fails in the description of caviton dynamics (the system does not reduce to the Korteveg-de Vries equation for a vanishing coupling).

A one-dimensional model for the propagation of a wavepacket of electrostatic (Langmuir) polarised waves in a uniform warm-electron-cold-ion plasma, including the nonlinear coupling, the nonlinear dynamical effects and dispersion, after rescaling, can be written in dimensionless variables τ (slow time) and ξ (comoving frame at the speed of sound for the ions):

$$\mathscr{E}_{\xi\xi}+(\lambda^2-\tfrac{1}{3}r)\mathscr{E}=0 \tag{5.1a}$$

$$2r_\tau+2rr_\xi+r_{\xi\xi\xi}=-\frac{\partial}{\partial\xi}\int_{-\infty}^{+\infty}|\mathscr{E}|^2\,\mathrm{d}\omega. \tag{5.1b}$$

We shall see that $\mathscr{E}(\xi,\tau,\lambda)$ represents the scaled slowly varying envelope component of the low-amplitude Langmuir wave of frequency ω, and r is related to the fractional change in the plasma density. The coupling term on the right-hand side of (5.1b) originates from the ponderomotive force and the integral comes from the fact that we consider a wavepacket (the integral is absent for a single wave). Finally, the parameter λ is proportional to the wavenumber.

First of all we remark that the system (5.1), while being similar to (3.11) (set $q=3r$, $2\partial_\tau=\partial_t$), is not integrable due to the different sign in front of the third-order derivative.

This is what we call a parametric (non-)integrability which may occur as soon as the 'nonlinear evolution' (5.1b) is coupled to the 'spectral problem' (5.1a) because then it is not possible to scale off all the constants.

We believe that this new type of nearly integrable system plays a very important role in physics and needs to be studied further, but this will be the subject of future work.

The derivation of (5.1) is standard [21-23] except for the fact that we consider a plasma wavepacket which modifies the expression of the pondermotive force. To make things clear we will now give necessary details about scales and dimensions.

The plasma wave (or Langmuir or space-charge or electrostatic wave) has the following dispersion relation [24]:

$$\omega^2=\omega_\mathrm{p}^2+3V_{\mathrm{Te}}^2k^2 \tag{5.2}$$

where V_{Te} is the thermal electron velocity and ω_{p} stands for the plasma frequency (of the electrons)

$$\omega_{\mathrm{p}}^2 = \omega_0^2(1+q_{\mathrm{e}}) \tag{5.3}$$

$$\omega_0^2 = 4\pi e^2 n_0/m_{\mathrm{e}}. \tag{5.4}$$

The fractional change q_{e} in the electron density of average value n_0 is a function of space and varies slowly in time.

We consider a linearily polarised electrostatic ($\boldsymbol{B}=0$) field vector $\boldsymbol{E}=(0, 0, E(z, t))$, propagating along the z direction. The equation of propagation resulting from (5.2) is

$$\left(\frac{\partial^2}{\partial t^2} - 3V_{\mathrm{Te}}^2\frac{\partial^2}{\partial z^2}\right)E(z, t) = -\omega_0^2(1+q_{\mathrm{e}}(z, \tau))E(z, t) \tag{5.5}$$

where the slow time τ will be defined below in (5.8).

Moreover, we consider E to be a wavepacket with slowly varying envelope component $\tilde{E}(\omega, z, \tau)$ (we work throughout with the complex quantity E):

$$E(z, t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \tilde{E}(\omega, z, \tau)\, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{5.6}$$

The small parameter in our problem is the ratio of the electron and ion masses, more precisely we set

$$\varepsilon = \left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \tag{5.7}$$

and the (slow) scaled time is chosen to be

$$\tau = \omega_0 \varepsilon^{5/2} t. \tag{5.8}$$

We shall work, moreover, in the comoving frame at the speed of sound for the ions

$$c_{\mathrm{s}} = \left(\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{m_{\mathrm{i}}}\right)^{1/2} \equiv \varepsilon\omega_0\lambda_{\mathrm{D}} \tag{5.9}$$

with the new space variable

$$\xi = \varepsilon^{1/2}\left(\frac{1}{\lambda_{\mathrm{D}}}z - \varepsilon\omega_0 t\right) \tag{5.10}$$

where λ_{D} is the electron Debeye wavelength

$$\lambda_{\mathrm{D}}^2 = \frac{K_{\mathrm{B}}T_{\mathrm{e}}}{4\pi n_0 e^2}. \tag{5.11}$$

The equation obtained from (5.5) can be written

$$\frac{\partial^2}{\partial\xi^2}\mathscr{E} + (\lambda^2 - \alpha^2 r)\mathscr{E} = \mathrm{O}(\varepsilon^{1/2}) \tag{5.12}$$

in which $\mathscr{E}$, r and the parameter λ are defined from expansions in powers of ε given in (5.27) below for $\mathscr{E}=\mathscr{E}^{(1)}$ and $r=q_{\mathrm{e}}^{(1)}$, and for λ;

$$\alpha^2\frac{\omega^2-\omega_0^2}{\omega_0^2} = \varepsilon\lambda^2. \tag{5.13}$$

This means that $\tilde{E}(\omega)$ has significant values only in a neighbourhood of ω_0. Finally, the constant α is given by

$$\alpha^2 = \frac{\omega_0^2 \lambda_D^2}{3V_{Te}^2} \equiv \frac{1}{3} \qquad \text{for } V_{Te}^2 = \frac{k_B T_e}{m_e} \tag{5.14}$$

which is precisely the thermal velocity of electrons.

The equation (5.11) is to first order the differential equation (5.1a), which we must complete with some asymptotic boundary conditions, chosen here to be

$$\mathscr{E}(\lambda, \xi, \tau) \exp(\mathrm{i}\lambda\xi) \to a(\lambda) \qquad \text{as } \xi \to \infty. \tag{5.15}$$

The wavepacket profile $a(\lambda)$ is a real bounded function of λ vanishing for $|\lambda| \to \infty$. The above choice is natural since it implies that, far from the interaction region (i.e. for $q_e \to 0$), the electrostatic field becomes a superposition of plane waves. Indeed, in the original variables and for the chosen dimensions (see (5.25) below) the behaviour (5.15) is (for fixed t)

$$\tilde{E}(\omega, z, t) \to \varepsilon a \exp[-\mathrm{i}(kz - c_s t)]\left(\frac{2m_e\omega^2 K_B T_e}{e^2}\right)^{1/2} \qquad z \to \infty. \tag{5.16}$$

In the following we will choose the complex λ plane to be the sheet of the two-fold Riemann surface defined by the following determination of the square root:

$$\varepsilon^{1/2}\lambda = \left(\frac{\omega^2 - \omega_0^2}{3\omega_0^2}\right)^{1/2} \operatorname{sgn}[\operatorname{Re}(\omega)] \equiv k\frac{V_{Te}}{\omega_0} \tag{5.17}$$

with $\operatorname{Re}(x)^{1/2} \geqslant 0$ for any quantity x, and $(-1)^{1/2} = \mathrm{i}$.

In order to obtain the fluid equations for the plasma, we first evaluate the low-frequency effect of the high-frequency field E in (5.6) on the electrons (ponderomotive force). The first-order variation of the field E around the average position z is $\delta z\, \partial E/\partial z$ where δz is the displacement due to the high-frequency electrostatic field: $m_e(\delta z)_{tt} = eE$.

After integration, we write (the effects of the slow variation of $\tilde{E}$ are considered separately)

$$\delta z = -\frac{e}{m_e}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \omega^{-2}\tilde{E}(\omega, z, \tau)\, \mathrm{e}^{-\mathrm{i}\omega t}. \tag{5.18}$$

The resulting ponderomotive force is obtained by taking the real part of the low-frequency terms in $\delta z\, \partial E/\partial z$, namely

$$f_p = \mathrm{e}\, \operatorname{Re}[\delta z\, \partial E/\partial z]_{LF} \tag{5.19}$$

which allows us to write

$$f_p = -\frac{1}{2}\frac{e^2}{m_e}\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \omega^{-2}|\tilde{E}(\omega, z, \tau)|^2. \tag{5.20}$$

To now write the fluid equation we apply the change of variable (5.8) and (5.10) in two steps

$$t \to t' = \left(\frac{m_e}{m_i}\right)^{1/2}\omega_0 t \to \tau = \varepsilon^{3/2}t' \qquad z \to z' = \frac{1}{\lambda_D}z \to \xi = \varepsilon^{1/2}(z' - t') \tag{5.21}$$

and we redefine the fields as

$$\tilde{E}' = \tilde{E}\left(\frac{e^2}{2m_e\omega^2 K_B T_e}\right)^{1/2} \qquad \varphi' = \frac{e}{K_B T_e}\varphi \qquad v_i' = v_i\left(\frac{m_i}{K_B T_e}\right)^{1/2} \tag{5.22}$$

where φ is the electrostatic potential and v_i the ion velocity. Using the fact that $m_i \gg m_e$ the momentum transfer equation for the electrons [24] becomes

$$\frac{\partial \varphi'}{\partial z'} - \frac{1}{1+q_e}\frac{\partial q_e}{\partial z'} - \frac{\partial}{\partial z'}\int_{-\infty}^{+\infty} d\omega |\tilde{E}'(\omega, z', \tau)|^2 = 0 \tag{5.23}$$

and for the ions

$$\frac{\partial v_i'}{\partial t'} + v_i' \frac{\partial v_i'}{\partial z'} = -\frac{\partial \varphi'}{\partial z'}. \tag{5.24}$$

The above system is completed with the continuity equation

$$\frac{\partial q_i}{\partial t'} + \frac{\partial}{\partial z'}[(1+q_i)v_i'] = 0 \tag{5.25}$$

and the Poisson equation

$$\frac{\partial^2 \varphi'}{\partial z'^2} = q_e - q_i. \tag{5.26}$$

Then after integration of (5.23) we go over to the variables (ξ, τ) and expand all functions in powers of ε as

$$\varphi' = \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + O(\varepsilon^3) \qquad \tilde{E}' = \varepsilon\mathscr{E}^{(1)} + O(\varepsilon^2) \tag{5.27}$$

and analogously for $q_{e,i}$ and v_i.

The system (5.23)-(5.26) then gives at order ε and ε^2:

$$q_e^{(1)} = \varphi^{(1)} \tag{5.28}$$

$$q_e^{(2)} = \varphi^{(2)} + \tfrac{1}{2}\varphi^{(1)2} - \int_{-\infty}^{+\infty} |\mathscr{E}^{(1)}|^2 \, d\omega \tag{5.29}$$

$$\partial_\xi q_i^{(1)} = \partial_\xi v_i^{(1)} \tag{5.30}$$

$$\partial_\tau q_i^{(1)} - \partial_\xi q_i^{(2)} + \partial_\xi v_i^{(2)} + \partial_\xi (q_i^{(1)} v_i^{(1)}) = 0 \tag{5.31}$$

$$\partial_\xi v_i^{(1)} = \partial_\xi \varphi^{(1)} \tag{5.32}$$

$$\partial_\tau v_i^{(1)} - \partial_\xi v_i^{(2)} + v_i^{(1)}\partial_\xi v_i^{(1)} = -\partial_\xi \varphi^{(2)} \tag{5.33}$$

$$q_e^{(1)} = q_i^{(1)} \tag{5.34}$$

$$\partial^2_{\xi\xi}\varphi^{(1)} = q_e^{(2)} - q_i^{(2)}. \tag{5.35}$$

The first consequence is that we may denote by r the following equal quantities:

$$r \doteq q_e^{(1)} = q_i^{(1)} = \varphi^{(1)} = v_i^{(1)} \tag{5.36}$$

which satisfies (5.28), (5.29), (5.32) and (5.34). Among the remaining equations, the second-order factors $q_i^{(2)}$, $q_e^{(2)}$, $v_i^{(22)}$, and $\varphi^{(2)}$ can be eliminated to obtain the following evolution equation for r (denoting $\mathscr{E}^{(1)}$ by $\mathscr{E}$):

$$2r_\tau + 2rr_\xi + r_{\xi\xi\xi} = -\frac{\partial}{\partial \xi}\int_{-\infty}^{+\infty} d\omega |\mathscr{E}|^2. \tag{5.37}$$

We have therefore derived the system (5.1) as a basic equation in the coupling of HF/LF waves in a plasma.

Another physical situation leads to the same kind of equation as (5.1), namely the 'nonlinear interaction between short and long capillary-gravity waves' [25] (see equation (4.1) of [25]). In that case it may be possible that a different situation (different value of the Weber number $W=(\rho/T)gh^2$, where ρ is the fluid density, T the surface tension and h the uniform depth) would lead to the integrable evolution equation (1.1).

Remark. If it is necessary to go to the variable defined in (5.12) and (5.13), one should first choose the expansion

$$\tilde{E}'=\varepsilon^{1/2}[\mathscr{E}^{(1)}+\mathrm{O}(\varepsilon)]$$

instead of that of (5.27), and then use in (5.37)

$$\mathrm{d}\omega=[3\varepsilon\omega_0\lambda+\mathrm{O}(\varepsilon^2)]\,\mathrm{d}\lambda.$$

Appendix. The Schrödinger scattering problem as a $\bar{\partial}$ problem

For completeness, we recall here the known result [1, 12, 13] that the following scattering problem:

$$\left(\frac{\partial^2}{\partial x^2}+\lambda^2-p(x)\right)f^{(\pm)}(\lambda,x)=0 \tag{A.1}$$

$$f^{(\pm)}(\lambda,x)\to \mathrm{e}^{\pm\mathrm{i}\lambda x}\qquad x\to\pm\infty \tag{A.2}$$

can be written as the $\bar{\partial}$ problem (2.1) and (2.2) with $r(\lambda)$ given by (2.10).

The system (A.1) and (A.2) is equivalent to the following integral equations:

$$\begin{aligned} f^{(-)}(\lambda,x)&=\mathrm{e}^{-\mathrm{i}\lambda x}+\frac{1}{\lambda}\int_{-\infty}^{x}\mathrm{d}y\,\sin\lambda(x-y)p(y)f^{(-)}(\lambda,y)\\ &=\mathrm{e}^{-\mathrm{i}\lambda x}+G_x^+f^-(\lambda,\cdot)\end{aligned} \tag{A.3}$$

$$\begin{aligned} f^{(+)}(-\lambda,x)&=\mathrm{e}^{-\mathrm{i}\lambda x}-\frac{1}{\lambda}\int_{x}^{+\infty}\mathrm{d}y\,\sin\lambda(x-y)p(y)f^{(+)}(-\lambda,y)\\ &=\mathrm{e}^{-\mathrm{i}\lambda x}+G_x^-f^{(+)}(\lambda,\cdot).\end{aligned} \tag{A.4}$$

If the real function $p(x)$ satisfies

$$\int_{-\infty}^{+\infty}\mathrm{d}x(1+|x|)|p(x)|<\infty \tag{A.5}$$

then the Volterra equation (A.3), (A.4) implies that $f^{(-)}(\lambda,x)$ (respectively $f^{(+)}(-\lambda,x)$) can be extended in the upper (respectively lower) half λ plane.

On the real axis we may compute

$$\begin{aligned} f^{(-)}(\lambda,x)-f^{(+)}(-\lambda,x)&=G_x^+f^{(-)}(\lambda,\cdot)-G_x^-f^{(+)}(\lambda,\cdot)\\ &\equiv(G_x^+-G_x^-)f^{(-)}(\lambda,\cdot)+G_x^-(f^{(+)}(\lambda,\cdot)-f^{(-)}(\lambda,\cdot))\\ &=\rho(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda x}-t(\lambda)\,\mathrm{e}^{-\mathrm{i}\lambda x}+G_x^-(f^{(+)}(\lambda,\cdot)-f^{(-)}(\lambda,\cdot))\end{aligned} \tag{A.6}$$

where we have defined

$$\rho(\lambda) \doteq \frac{1}{2i\lambda} \int_{-\infty}^{+\infty} dy\, e^{-i\lambda y} p(y) f^{(-)}(\lambda, y) \tag{A.7}$$

$$t(\lambda) \doteq \frac{1}{2i\lambda} \int_{-\infty}^{+\infty} dy\, e^{-i\lambda y} p(y) f^{(-)}(\lambda, y). \tag{A.8}$$

Adding the quantity $t(\lambda) f^{(-)}(\lambda, x)$ to both sides of equation (A.6), we get

$$[f^{(-)}(\lambda, x)(1+t) - f^{(+)}(-\lambda, x)] = \rho\, e^{i\lambda x} + G_x^-[f^{(-)}(\lambda, \cdot)(1+t) - f^{(+)}(-\lambda, \cdot)] \tag{A.9}$$

which may be compared with

$$\begin{aligned} \rho(\lambda) f^{(+)}(\lambda, x) &= \rho(\lambda)\, e^{i\lambda x} + \rho(\lambda) G_x^- f^{(+)}(\lambda, \cdot) \\ &= \rho(\lambda)\, e^{i\lambda x} + G_x^- \rho(\lambda) f^{(+)}(\lambda, \cdot). \end{aligned} \tag{A.10}$$

Assuming for simplicity that the homogeneous integral equation, with integral operator G_x^-, has only the vanishing solution, comparing (A.9) with (A.10) implies that

$$[1 + t(\lambda)] f^{(-)}(\lambda, x) - f^{(+)}(-\lambda, x) = \rho(\lambda) f^{(+)}(\lambda, x). \tag{A.11}$$

This equation can be written

$$\psi^+(\lambda, x) - \psi^-(\lambda, x) = \rho(\lambda) \psi^-(-\lambda, x) \tag{A.12}$$

or

$$\frac{\partial}{\partial \bar{\lambda}} \psi(\lambda, x) = \rho(\lambda) \psi(-\lambda, x) \delta^-(\lambda_I) \tag{A.13}$$

for the function ψ defined by

$$\begin{aligned} \psi(\lambda, x) &= [1 + t(\lambda)] f^{(-)}(\lambda, x) \qquad \operatorname{Im} \lambda > 0 \\ &= f^{(+)}(-\lambda, x) \qquad \operatorname{Im} \lambda < 0 \end{aligned} \tag{A.14}$$

and for δ^- defined in (2.11)†. The $\bar{\partial}$ equation (A.13) is equivalent to (2.1) with the definition (2.7) and the relation (2.10) between $r(\lambda, x)$ and $\rho(\lambda)$ (we have assumed no non-vanishing solution to the homogeneous equation, i.e. no bound states $\Rightarrow C_n = 0$ for all n).

Finally the λ behaviour (2.2) is a consequence of the definition of ψ and of the integral equations (A.3) and (A.4).

It remains to prove (2.13), which is obtained by comparing the behaviour (1.2)

$$u(\lambda, x) \to a(\lambda) \exp[-i\lambda(x - \lambda^2 t)] \qquad x \to \infty \quad \lambda \in \mathbb{R} \tag{A.15}$$

with the behaviour of $\psi(\lambda, x)$ for $\operatorname{Im} \lambda < 0$ resulting from (A.2):

$$\psi(\lambda - i0, x) = f^{(+)}(-\lambda, x) \to e^{-i\lambda x} \qquad x \to \infty \quad \lambda \in \mathbb{R}. \tag{A.16}$$

In this formalism the bound states are related to the N solutions of the homogeneous version of the integral equation (A.4), i.e.

$$f^{(+)}(-\lambda, x) = f_A^{(+)}(-\lambda, x) + \sum_1^N \frac{\varphi_n(x)}{\lambda - \lambda_n} \tag{A.17}$$

† Note that $1 + t(\lambda)$ is the transmission coefficient $T(\lambda)$ previously introduced in section 3.

where $f_A^{(+)}$ is the solution of (A.4), holomorphic in the lower half λ plane and where $\varphi_n(x)$ satisfies

$$\varphi_n(x) = -\frac{1}{\lambda_n}\int_x^{+\infty} \mathrm{d}y \sin\lambda_n(x-y)p(y)\varphi_n(y). \tag{A.18}$$

It is then possible to show that the discrete part to be added to (A.13) is indeed given in general by (2.10).

References

[1] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* (Amsterdam: North Holland)
[2] Kaup D J and Newell A C 1979 *Adv. Math.* **31** 67
[3] Leon J 1987 *Phys. Lett.* **123A** 65
[4] Leon J 1988 *J. Math. Phys.* **29** 2012
[5] Boiti M, Leon J, Martina L and Pempinelli F 1988 *J. Phys. A: Math. Gen.* **21** 3611
[6] Leon J 1988 *Phys. Lett.* **131A** 79
[7] Leon J and Pempinelli F 1989 *Nonlinear Evolution Equations: Integrability and Spectral Methods* ed A Degasperis, A P Fordy and M Lakshmanan (Manchester: Manchester University Press)
[8] Mel'nikov V K 1988 *Phys. Lett.* **133A** 493
[9] McCall S L and Hann E L 1969 *Phys. Rev.* **183** 457
Lamb G L Jr 1974 *Phys. Rev.* A **8** 422; 1975 *Phys. Rev.* A **12** 2052
Ablowitz M J, Kaup D J and Newell A C 1974 *J. Math. Phys.* **15** 1852
[10] Doolen G D, Dubois D F and Rose H A 1985 *Phys. Rev. Lett.* **54** 804
[11] Karpman V I 1975 *Phys. Scr.* **11** 263
Kaw P K and Nishikawa K 1975 *J. Phys. Soc. Japan* **38** 1753
Kaup D J 1987 *Phys. Rev. Lett.* **59** 2062
[12] Jaulent M, Manna M and Martinez-Alonzo L 1988 *Inverse Problems* **4** 123
[13] Jaulent M and Manna M 1987 *J. Math. Phys.* **28** 2338
[14] Manakov S V 1982 *Sov. Phys.-JETP* **56** 37
[15] Goldman M V 1984 *Rev. Mod. Phys.* **56** 709
[16] Zakharov V E 1972 *Sov. Phys.-JETP* **35** 908 [1972 *Zh. Eksp. Teor. Fiz.* **62** 1745]
[17] Chen H H and Liu C S 1977 *Phys. Rev. Lett.* **39** 1147
[18] Laedke E W and Spatschek K H 1980 *Phys. Fluids* **23** 44
[19] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
[20] Kaup D J 1987 *Phys. Rev. Lett.* **59** 2062
[21] Kaw P K and Nishikawa K 1975 *J. Phys. Soc. Japan* **38** 1753
[22] Nishikawa K, Hojo H, Nima K and Ikezi H 1974 *Phys. Rev. Lett.* **33** 148
[23] Makhankov V G 1974 *Phys. Lett.* **50A** 42
[24] Krall N A and Trivelpiece A W 1973 *Principles of Plasma Physics* (New York: McGraw-Hill)
[25] Kawahara T, Sugimoto N and Kakutani T 1975 *J. Phys. Soc. Japan* **39** 1379